On the stability of a shear layer

By J. MENKES
California Institute of Technology, Jet Propulsion Laboratory

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The effects of density variation in the absence of gravity on the stability of a horizontal shear layer between two streams of uniform velocities is investigated. The density is assumed to decrease exponentially with height and the velocity is represented by $U(y) = \tanh y$.

The method of small disturbances is employed to obtain the neutral stability curve. It is demonstrated that disturbances with wave-numbers larger than the width of the transition layer are attenuated.

Qualitative agreement with experimental evidence is obtained.

1. Introduction

The mixing zone between two parallel streams, each of which has initially uniform velocity and density, may be represented for the purpose of stability analysis by an inviscid shear layer. The general problem of stability of an inviscid fluid with continuously varying velocity and density distribution in a direction normal to the mean flow was first attacked by Taylor (1931) and Goldstein (1931). Employing the method of small disturbances, they obtained an equation of the Orr–Sommerfeld type. While the equation is linear, its coefficients depend on the velocity and density distribution in the unperturbed shear layer. In order to render the problem mathematically tractable, they considered simple flows in which the velocity or velocity gradient is constant, and the density is constant or varies exponentially. The properties of the more general layer were to be deduced from a superposition of the simple flows. Drazin (1958) observed that a velocity distribution that varies as the hyperbolic tangent of the transverse co-ordinate can be handled analytically; such a profile does represent the transition of velocity rather well. Blackshear (1957) recognized the connexion between the stability of a flame downstream of a bluff body flame stabilizer and the investigations of Taylor and Goldstein. Another interesting application of the theory is concerned with the stability of a liquid film on the ablating body of a vehicle re-entering the atmosphere. Here we investigate the stability of a free, heterogeneous shear layer, neglecting the influence of gravity.

2. Derivation of the stability equation

The equations of motion governing the behaviour of an inviscid fluid under the action of gravity (which is included initially to make the comparison with the work of Taylor, Goldstein and Drazin more convenient) are Euler's equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} - g\nabla y, \tag{1}$$

the condition of incompressibility $\frac{D\rho}{Dt} = 0,$ (2)

and the equation of continuity $\nabla . \mathbf{u} = 0.$ (3)

The velocity components, the pressure and the density are assumed to consist of a time-independent part and a perturbation, i.e.

$$u = V[U(y) + \psi'_y(x,y,t)], \quad v = -V\psi'_x(x,y,t), \tag{4a}$$

$$p = P(y) + p'(x,y,t), \quad \rho = \rho_0[\bar{\rho}(y) + \rho'(x,y,t)], \tag{4b}$$

$$\psi' = \psi(y) \exp[ik(x-ct)], \quad c = c_r + ic_i. \tag{4c}$$

Here, $V = U(\infty) = -U(-\infty)$, and ψ' is a perturbation stream function. $U(y)$, $P(y)$ and $\bar{\rho}(y)$ describe the ambient state whose stability is to be investigated. $x = x_1/d$, $y = y_1/d$, where x_1, y_1 are the physical co-ordinates and d is so chosen that $dU/dy = 1$ at $y = 0$; thus d characterizes the width of the transition layer. Denoting the Froude number V^2/gd by F and eliminating ρ' and p' from equations (1), (2) and (3) we obtain

$$(U-c)(\psi'' - k^2\psi) - U''\psi + (\ln\bar{\rho})'[(U-c)\psi' - (U-c)'\psi] - \frac{(\ln\bar{\rho})'\psi}{F(U-c)} = 0, \tag{5}$$

where primes denote differentiation with respect to y. We now set $\bar{\rho} = \exp(-2Ly)$ and obtain:

$$(U-c)(\psi'' - k^2\psi) - U''\psi - 2L[(U-c)\psi' - (U-c)'\psi] + \frac{2L}{F}\frac{\psi}{U-c} = 0. \tag{6}$$

It is well known that $J = 2L/F$ (the Richardson number) measures the ratio of the buoyancy forces to the inertia forces. In meteorological applications, where the characteristic length associated with the phenomena is usually large, buoyancy forces are very important. In the kind of small-scale aerodynamic application that is under consideration here, the effect of gravity is usually negligible. The parameter L measures the degree of heterogeneity of inertia of the fluid. The density gradient through a flame is very sharp, and it is expected that this will have a considerable effect on the stability of the flow system. It is thus proposed to neglect the term in the equation that is multiplied by the Richardson number and keep the one that has L as its coefficient. The present investigation consequently endeavours to deal with the limiting case $J = 0$, and $L \neq 0$.

3. Analysis

Drazin (1958) recognized that a velocity distribution of the form $U = \tanh y$ can be studied analytically if the independent variable y is changed to $U(y)$. With the aid of this transformation (which is essentially a hodograph transformation), the stability equation is cast into a special form of the Lamé equation. By a transformation of the dependent variable suggested by the method of Papperitz as applied to the hypergeometric equation (e.g. see Morse & Feshbach 1953, p. 539), equation (6) is cast into a form which under certain circumstances permits one to obtain a trivial integral, namely, a constant. Only the degenerate case $c = 0$ is treated by Drazin.

As the primary velocity distribution we take $U(y) = \tanh y$, and we introduce U as the independent variable. Noting that

$$dU/dy = \operatorname{sech}^2 y = 1-U^2, \quad d^2U/dy^2 = -2U(1-U^2),$$

denoting now by primes differentiation with respect to U, and setting

$$\psi(y) = \phi(U),$$

one obtains: $$\phi'' - 2\frac{(L+U)}{1-U^2}\phi' + \left[\frac{2(U+L)}{(U-c)(1-U^2)} - \frac{k^2}{(1-U^2)^2}\right]\phi = 0 \tag{7}$$

with the boundary conditions $k\phi(U) = 0$ at $U = \pm 1$.

It is instructive at this point to set $L = c = 0$, thereby obtaining

$$(1-U^2)\phi'' - 2U\phi' + \left[2 - \frac{k^2}{1-U^2}\right]\phi = 0. \tag{8}$$

The solution of equation (8), consistent with our boundary conditions, is the Legendre function $P_1^k(U)$, where

$$P_1^0 \equiv P_1 = U \quad \text{and} \quad P_1^1 = (1-U^2)^{\frac{1}{2}}.$$

These are the *only* two solutions, since $k \leqslant 1$ when $L = 0$.

Equation (7) is of a rather simple type. Its singularities which are located at ± 1 and c are regular singularities. It can be demonstrated that the point at infinity is also a regular singularity. The substitution into equation (7) of

$$Z = (U-1)^{-\alpha_1}(U-c)^{-\alpha_2}(U+1)^{-\alpha_3}\phi,$$

where the α's are each one of the exponents relative to the singularity and are given by

$$\alpha_1 = \frac{L}{2}(\lambda-1), \quad \alpha_2 = 0, \quad \alpha_3 = \frac{L}{2}(\lambda+1); \quad \lambda = +\sqrt{\left(1+\frac{k^2}{L^2}\right)},$$

leads to the equation

$$Z'' + \left(\frac{1+\lambda L}{U-1} + \frac{1+\lambda L}{U+1}\right)Z' + \frac{1}{(U+1)(U-1)(U-c)} \times [(L^2\lambda^2 + L\lambda - 2)U - (L^2\lambda^2 c + L\lambda c + 2L)]Z = 0,$$

with the boundary conditions that Z be *regular* at $U = c$ and ± 1.

The most general solution of equation (9) can be written in the symbolic form of the Riemann P-function:

$$Z = P\begin{bmatrix} +1 & c & -1 & \infty & \\ 0 & 0 & 0 & \sigma & U \\ a_1 & a_2 & a_3 & \tau & \end{bmatrix}.$$

The fact that one each of the exponents relative to the singularities at $+1$, -1, and c vanishes implies that *one* of the solutions is regular there. For the case in which

$$L^2\lambda^2 + L\lambda - 2 = 0$$

and $$(L^2\lambda^2 c + L\lambda c + 2L) = 0 \quad (L \neq 0),$$

one of the exponents at infinity vanishes as well, thereby assuring a solution that is regular everywhere. The only such function is, of course, a constant. A sufficient condition for the existence of the solution, $Z = \text{const.}$, is

$$L\lambda = 1 \text{ or } -2 \quad \text{and} \quad c = -L.$$

The negative root for $L\lambda$ must be discarded because the exponent at $U = -1$ must be positive in order to satisfy the boundary conditions there. The fact that c is negative means that the waves propagate in phase with the heavier and lower ($y < 0$) fluid rather than in the upper and lighter layers. The neutral stability boundary is thus given by $L\lambda = 1$, i.e. by

$$L^2 + k^2 = 1.$$

4. Results and discussion

The form of the neutral stability curve warrants a few words of explanation. The circle $L^2 + k^2 = 1$ corresponds to the *upper* branch of the stability boundary; the origin represents the degenerate *lower* branch of the stability boundary. The two 'branches' do actually join up in the complex domain. Thus the area contained between the two 'branches' represents unstable disturbances.

Disturbances having a wavelength less than the characteristic length d are stable. The effect of the density gradient is stabilizing. The latter conclusion is in qualitative agreement with the known fact that a cooled wall has a stabilizing effect on a boundary layer. A simple illustration of the effect of the density gradient may be had by the following considerations. Let us consider the case of self-excited disturbances. This corresponds to $c_i \neq 0$ and consequently $U - c$ does not vanish in the real domain. Neglect the Froude number in (7) and multiply it by the complex conjugate $\tilde{\psi}$ of ψ. If one now subtracts from the resultant equation *its* complex conjugate one obtains (the method is essentially the same as Lin's (1955)):

$$\frac{d}{dy}[\psi'\tilde{\psi} - \tilde{\psi}'\psi] = \frac{c - \tilde{c}}{|U - c|^2}(U'' + U'\bar{\rho}'/\bar{\rho})\,|\psi|^2,$$

or

$$\frac{d}{dy}[\psi'\tilde{\psi} - \tilde{\psi}'\psi] = \frac{2ic_i\,|\psi|^2}{\bar{\rho}|U - c|^2}\frac{d}{dy}[\bar{\rho}U'].$$

When both sides are integrated from $-\infty$ to $+\infty$ the left-hand side vanishes on account of the boundary conditions and we are left with the following generalization of Rayleigh's criterion:

$$2ic_i \int_{-\infty}^{+\infty} \frac{|\psi|^2}{\bar{\rho}\,|U - c|^2}[\bar{\rho}U']'\,dy = 0.$$

This integral illustrates what we have set out to show, namely, that for an inhomogeneous fluid the fact that U'' vanishes somewhere is *not* the proper criterion for the existence of amplified disturbances. The integral also shows that the quantity $d(\rho dU/dy)/dy$ which was shown by Lees & Lin (1946) to dominate the behaviour of the inertial forces in the compressible (i.e. non-homogeneous) boundary layer arises naturally in the present criterion. There is, however, an essential difference between the free shear layer and a boundary layer-type shear flow. The free shear layer is unstable for a band of disturbances even in the inviscid case (corresponding to infinite Reynolds number), whereas the 'inviscid' boundary layer is absolutely stable.

The fact that only disturbances within a certain band of wavelengths are amplified has been demonstrated experimentally by Blackshear. While it must be borne in mind that his experimental configuration is not the best approximation to the mathematical model investigated here, it is of interest to note that he found that disturbances with a non-dimensional wave-number of order unity or larger are stable. The effect of the density gradient is also qualitatively confirmed by his experiment.

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